

CLASSIFICATION OF DOUBLE FLAG VARIETIES OF COMPLEXITY 0 AND 1

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ABSTRACT. A classification of double flag varieties of complexity 0 and 1 is obtained. An application of this problem to decomposing tensor products of irreducible representations of semisimple Lie groups is considered.

1. INTRODUCTION

Let G be a semisimple complex Lie group, B a Borel subgroup of G . Suppose that the group G acts on an irreducible complex algebraic variety X . This action induces an action of B on X .

Definition 1. The codimension of a generic B -orbit in X is called the complexity of an action $G : X$ and is denoted by $c(X) = c_G(X)$.

Remark. By the Rosenlicht theorem, $c(X) = \text{tr.deg } \mathbb{C}(X)^B / \mathbb{C}$.

A subgroup $P \subseteq G$ is parabolic if P contains a Borel subgroup. Suppose P and Q are parabolic subgroups. The variety $X = G/P \times G/Q$ is called a double flag variety. This paper is devoted to classification of double flag varieties of complexity at most 1. Littelmann [1] classified double flag varieties of complexity 0 for maximal parabolic subgroups. Stembridge [2] classified all double flag varieties of complexity 0. Panyushev [3] found complexities of double flag varieties for all maximal parabolic subgroups. In this paper we obtain these already known results by a uniform method and complete the classification in the case of complexity 1.

The problem of classifying double flag varieties of complexity 0 and 1 has an application to decomposing tensor products of irreducible representations of G into irreducible summands. Any irreducible G -module can be realized as the space of global sections for some line bundle \mathcal{L} over G/P (here P is parabolic). We may regard the tensor product of the spaces of sections $H^0(G/P, \mathcal{L}) \otimes H^0(G/Q, \mathcal{M})$ as the space of sections of a line bundle over the product of varieties G/P and G/Q , i.e., $H^0(G/P \times G/Q, \mathcal{L} \boxtimes \mathcal{M})$. Here $\mathcal{L} \boxtimes \mathcal{M} \rightarrow G/P \times G/Q$ is a line bundle such that the fibre $(\mathcal{L} \boxtimes \mathcal{M})_{(x,y)}$ over the point (x, y) is the tensor products of the fibers \mathcal{L}_x and \mathcal{M}_y over the points $x \in G/P$ and $y \in G/Q$. If the complexity of the variety $X = G/P \times G/Q$ equals 0 or 1, then there exists an effective method to decompose the space of sections $H^0(X, \mathcal{N})$ of a line bundle $\mathcal{N} \rightarrow X$ into irreducible submodules.

Suppose a semisimple group is decomposed into almost direct product of simple subgroups: $G = G_1 \cdot \dots \cdot G_s$. Then parabolic subgroups $P, Q \subseteq G$ are decomposed into almost direct products of parabolic subgroups $P_i, Q_i \subseteq G_i$. We have $c_G(G/P \times G/Q) = c_{G_1}(G_1/P_1 \times G_1/Q_1) + \dots + c_{G_s}(G_s/P_s \times G_s/Q_s)$. So the problem of computing complexity of double flag varieties for semisimple groups reduces to the same problem for simple groups.

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Suppose G is a classical matrix group; then we assume that B consists of upper triangular matrices (here we assume that the group G preserves a bilinear form with an antidiagonal matrix in orthogonal and symplectic cases). Then parabolic subgroups containing B have a block-triangular structure and are determined by the sizes of diagonal blocks. A group SO_n for even n is an exception. For this group not all parabolic subgroups have this form. The remaining parabolics are transformed to the described form by conjugation with transposition of two middle basic vectors (the diagram automorphism). We mark such parabolic subgroups with strokes.

In case of exceptional groups parabolic subgroups are determined by a subset $\Pi \setminus I$ of the set of simple roots Π , where $I \subseteq \Pi$ is the system of simple roots of a standard Levi subgroup. Simple roots are numbered as in [4].

In this paper we prove the following classification theorems.

Theorem 1. *Let G be a classical matrix group (SL_n, SO_n, Sp_n) . Then all double flag varieties of complexity 0 and 1 correspond to the pairs of parabolic subgroups given in Tables 1, 2, 3 (the classification is given up to permutation of parabolics in a pair in all cases, up to simultaneous transposition with respect to the secondary diagonal for SL_n , and up to the diagram automorphism for SO_{2n}).*

	complexity 0		complexity 1	
number of blocks in P and Q	P	Q	P	Q
2,2	(p_1, p_2)	(q_1, q_2)		
2,3	(p_1, p_2) (p_1, p_2) $(2, p_2)$	$(1, q_2, q_3)$ $(q_1, 1, q_3)$ (q_1, q_2, q_3)	$(3, p_2), p_2 \geq 3$ $(p_1, p_2), p_1, p_2 \geq 3$ $(p_1, p_2), p_1, p_2 \geq 3$	$(q_1, q_2, q_3), q_1, q_2, q_3 \geq 2$ $(2, 2, q_3), q_3 \geq 2$ $(2, q_2, 2), q_2 \geq 2$
2,4			$(2, p_2)$ $(p_1, p_2), p_1, p_2 \geq 2$ $(p_1, p_2), p_1, p_2 \geq 2$	(q_1, q_2, q_3, q_4) $(1, 1, 1, q_4)$ $(1, 1, q_3, 1)$
2, s	$(1, p_2)$	(q_1, q_2, \dots, q_s)		
3,3			$(1, 1, p_3)$ $(1, p_2, 1)$	(q_1, q_2, q_3) (q_1, q_2, q_3)

TABLE 1. pairs of parabolic subgroups corresponding to double flag varieties of complexity 0 and 1 for SL_n

Theorem 2. 1) *There are no double flag varieties of complexity 0 and 1 for the groups G_2 , F_4 and E_8 .*

2) *For E_6 , the varieties of complexity 0 correspond to the following pairs of parabolic subgroups:*

$(\{\alpha_1\}, \{\alpha_1\}), (\{\alpha_1\}, \{\alpha_2\}), (\{\alpha_1\}, \{\alpha_4\}), (\{\alpha_1\}, \{\alpha_5\}), (\{\alpha_1\}, \{\alpha_6\}), (\{\alpha_2\}, \{\alpha_5\}),$
 $(\{\alpha_4\}, \{\alpha_5\}), (\{\alpha_5\}, \{\alpha_5\}), (\{\alpha_5\}, \{\alpha_6\}), (\{\alpha_1\}, \{\alpha_1, \alpha_5\}), (\{\alpha_5\}, \{\alpha_1, \alpha_5\});$

the varieties of complexity 1 correspond to the following pairs of parabolic subgroups:

$(\{\alpha_1\}, \{\alpha_1, \alpha_2\}), (\{\alpha_1\}, \{\alpha_1, \alpha_6\}), (\{\alpha_1\}, \{\alpha_4, \alpha_5\}), (\{\alpha_1\}, \{\alpha_5, \alpha_6\}), (\{\alpha_5\}, \{\alpha_1, \alpha_2\}),$
 $(\{\alpha_5\}, \{\alpha_1, \alpha_6\}), (\{\alpha_5\}, \{\alpha_4, \alpha_5\}), (\{\alpha_5\}, \{\alpha_5, \alpha_6\}).$

3) *For E_7 , the varieties of complexity 0 correspond to the following pairs of parabolic subgroups:*

	complexity 0		complexity 1	
number of blocks in P and Q	P	Q	P	Q
2,2	(p, p) (p, p)	(p, p) $(p, p)'$		
2,3	(p, p) (p, p)	$(q_1, q_2, q_1), q_1 \leq 3$ $(q, 2, q)$	$(6, 6)$	$(4, 4, 4)$
2,4	(p, p) (p, p) $(4, 4)$	$(1, q, q, 1)$ $(1, q, q, 1)'$ $(2, 2, 2, 2)'$	$(4, 4)$ $(5, 5)$ $(5, 5)$ $(5, 5)$ $(5, 5)$	$(2, 2, 2, 2)$ $(2, 3, 3, 2)$ $(3, 2, 2, 3)$ $(2, 3, 3, 2)'$ $(3, 2, 2, 3)'$
2,5	(p, p)	$(1, 1, q, 1, 1)$	$(4, 4)$ $(4, 4)$	$(1, 2, 2, 2, 1)$ $(2, 1, 2, 1, 2)$
2,6			$(4, 4)$ $(4, 4)$	$(1, 1, 2, 2, 1, 1)$ $(1, 1, 2, 2, 1, 1)'$
3,3	$(1, p, 1)$ $(p, 1, p)$	(q_1, q_2, q_1) $(p, 1, p)$	$(2, 2, 2)$ $(2, p, 2), p > 1$	$(2, 2, 2)$ $(q, 1, q)$
3,4	$(1, p, 1)$	(q_1, q_2, q_2, q_1)	$(2, 2, 2)$	$(1, 2, 2, 1)$
3,5			$(1, p, 1)$ $(2, 1, 2)$	$(q_1, q_2, q_3, q_2, q_1)$ $(1, 1, 1, 1, 1)$
3,6			$(1, p, 1)$	$(q_1, q_2, q_3, q_3, q_2, q_1)$
4,4			$(1, 2, 2, 1)$ $(1, 2, 2, 1)$	$(1, 2, 2, 1)$ $(1, 2, 2, 1)'$

TABLE 2. pairs of parabolic subgroups corresponding to double flag varieties of complexity 0 and 1 for SO_n

	complexity 0		complexity 1	
number of blocks in P and Q	P	Q	P	Q
2,2	(p, p)	(p, p)		
2,3	(p, p)	$(1, q, 1)$	(p, p)	$(2, q, 2)$
2,4			$(2, 2)$	$(1, 1, 1, 1)$
3,3	$(1, p, 1)$	(q_1, q_2, q_1)		
3,4			$(1, p, 1)$	(q_1, q_2, q_2, q_1)
3,5			$(1, p, 1)$	$(q_1, q_2, q_3, q_2, q_1)$

TABLE 3. pairs of parabolic subgroups corresponding to double flag varieties of complexity 0 and 1 for Sp_n

$(\{\alpha_1\}, \{\alpha_1\}), (\{\alpha_1\}, \{\alpha_6\}), (\{\alpha_1\}, \{\alpha_7\});$

the varieties of complexity 1 correspond to the following pairs of parabolic subgroups:

$(\{\alpha_1\}, \{\alpha_2\}).$

The paper is organized as follows. In Section 2, we discuss a method of decomposing the space of sections $H^0(X, \mathcal{N})$ of a line bundle $\mathcal{N} \rightarrow X$ into irreducible submodules whenever the complexity of X equals 0 or 1. Some examples of decomposing tensor products of irreducible representations using this method are considered. In Section 3, some general theorems concerning complexity of double

flag varieties are given. In Sections 4 and 5, we obtain the classification of double flag varieties of complexity 0 and 1 for classical and exceptional groups, respectively.

2. DECOMPOSITION OF SPACES OF SECTIONS

Let G act on a normal variety X . We consider prime B -stable divisors on X . To each prime divisor D we assign a homomorphism $\text{ord}_D : \mathbb{C}(X)^\times \rightarrow \mathbb{Z}$.

Any line bundle over X can be G -linearized [5]. Any Cartier divisor δ is linearly equivalent to a B -stable divisor; this can be proved by choosing a B -semi-invariant rational section of the line bundle $\mathcal{O}(\delta)$ [6].

2.1. Case of complexity 0. In this case we have $\mathbb{C}(X)^B = \mathbb{C}$. Therefore any B -semi-invariant function is uniquely determined by its weight up to a scalar multiple. The value $\text{ord}_D(f)$ does not change if we multiply f by a constant. Thus we can map (in general, not injectively) the set of B -stable prime divisors to the group $\text{Hom}(\Lambda, \mathbb{Z})$, where $\Lambda = \Lambda(X)$ is the lattice of eigenweights of B -semi-invariant rational functions on X . B -stable divisors can be regarded as vectors in $\text{Hom}(\Lambda, \mathbb{Z})$; hence $\text{ord}_D f_\lambda = \langle v_D, \lambda \rangle$, where $v_D \in \text{Hom}(\Lambda, \mathbb{Z})$ is the vector corresponding to D and f_λ is a function of weight λ .

There are finitely many prime B -stable divisors, since they lie in the complement of the open B -orbit.

Denote by V_λ an irreducible G -module of highest weight λ . Denote by λ^* the highest weight of the dual module. A map $\lambda \mapsto \lambda^*$ can be extended to all weights by linearity.

Now we formulate the main theorem about decomposing spaces of sections.

Theorem 3 ([7]). *Let X be a variety of complexity 0 and $\delta = \sum m_i D_i$ a Cartier divisor, where D_i are the distinct B -stable divisors on X . Then*

$$H^0(X, \mathcal{O}(\delta)) \simeq \bigoplus_{\lambda \in \mathcal{P}(\delta) \cap \Lambda} V_{\lambda + \pi(\delta)},$$

where $\pi(\delta)$ is the weight of the canonical section s_δ corresponding to the divisor δ and

$$\mathcal{P}(\delta) = \{\lambda \in \Lambda \otimes \mathbb{Q} \mid \langle v_i, \lambda \rangle \geq -m_i, \forall i\}$$

is a polytope in $\Lambda \otimes \mathbb{Q}$, where v_i are the vectors corresponding to D_i .

Proof. One of the equivalent definitions of spherical variety is that for any G -line bundle $\mathcal{L} \rightarrow X$ the action $G : H^0(X, \mathcal{L})$ is multiplicity-free. (By definition, a variety is spherical if its complexity equals zero.)

Thus it is sufficient to describe the set of highest weights. A B -semi-invariant section can be represented as $s = f_\lambda s_\delta$. The condition that the divisor $\text{div } s = \text{div } f_\lambda + \delta$ is effective is equivalent to $\lambda \in \mathcal{P}(\delta) \cap \Lambda$. \square

2.2. Case of complexity 1. For varieties of complexity 1 the theory is a bit more complicated. Suppose for simplicity that X is a rational variety. Then by the Lüroth theorem we have $\mathbb{C}(X)^B \simeq \mathbb{C}(\mathbb{P}^1)$. Therefore B -semi-invariant functions are determined by their weights uniquely up to multiplication by a function from $\mathbb{C}(X)^B \simeq \mathbb{C}(\mathbb{P}^1)$, i.e., a B -semi-invariant function can be represented as $f_\lambda q$, where f_λ is a fixed function of weight λ and $q \in \mathbb{C}(\mathbb{P}^1)$. There is a rational map $X \dashrightarrow \mathbb{P}^1$ whose general fibres are the closures of general B -orbits. Therefore we can describe prime B -stable divisors as follows. Except a finite number of them, prime B -stable divisors form a family parameterized by the projective line except a finite number of points.

Similar to the case of complexity 0 we can associate a vector $v_D \in \text{Hom}(\Lambda, \mathbb{Z})$ to any B -stable prime divisor D restricting ord_D to $\{f_\lambda \mid \lambda \in \Lambda\}$ (here we assume that

the map $\lambda \mapsto f_\lambda$ is a group homomorphism). By restricting ord_D to $\mathbb{C}(X)^B \simeq \mathbb{C}(\mathbb{P}^1)$ we obtain a valuation of $\mathbb{C}(\mathbb{P}^1)$ with center $z_D \in \mathbb{P}^1$ and order $h_D \in \mathbb{Z}_+$ of a local coordinate at z_D (if $h_D = 0$, we can take any point from \mathbb{P}^1 for z_D). Then $\text{ord}_D f = \langle v_D, \lambda \rangle + h_D \text{ord}_{z_D} q$. Thus we associate with D a triple (v_D, z_D, h_D) . Remove sufficiently many points from the projective line. Then we may assume that for remaining points in \mathbb{P}^1 there exists a unique prime B -stable divisor D such that $z_D = z$, and furthermore, $v_D = 0$, $h_D = 1$.

For varieties of complexity 1 there is a similar theorem about decomposition of the space of sections:

Theorem 4 ([8]). *Let X be a rational variety of complexity 1 and $\delta = \sum m_i D_i$ a Cartier divisor, where the sum ranges over all B -stable prime divisors on X (we assume that only finitely many m_i are nonzero). Then*

$$H^0(X, \mathcal{O}(\delta)) \simeq \bigoplus_{\lambda \in \mathcal{P}(\delta) \cap \Lambda} m(\delta, \lambda) V_{\lambda + \pi(\delta)},$$

where $\pi(\delta)$ is the weight of the canonical section s_δ corresponding to the divisor δ ,

$$\mathcal{P}(\delta) = \{\lambda \in \Lambda \otimes \mathbb{Q} \mid \langle v_i, \lambda \rangle \geq -m_i, \forall i, \text{ whenever } h_i = 0\},$$

where (v_i, z_i, h_i) is the triple corresponding to the divisor D_i , and the multiplicity $m(\delta, \lambda)$ of the module $V_{\lambda + \pi(\delta)}$ in the decomposition equals

$$m(\delta, \lambda) = \max \left(1 + \sum_{z \in \mathbb{P}^1} m_z, 0 \right),$$

$$\text{where } m_z = \min_{z_i = z, h_i \neq 0} \left\lceil \frac{\langle v_i, \lambda \rangle + m_i}{h_i} \right\rceil, \forall z \in \mathbb{P}^1.$$

The proof of this theorem is similar to the proof of Theorem 3.

2.3. Examples.

Example 1. Let $G = Sp_n$, $n = 2l$. Consider the double flag variety $X = G/P \times G/Q$ corresponding to the pair $(1, 2l - 2, 1)$, (l, l) of parabolic subgroups. This is a variety of complexity 0. Suppose e_1, \dots, e_n is the standard basis of \mathbb{C}^n , ϵ_i are the weights of the vectors e_i with respect to the diagonal maximal torus T , $\omega_i = \epsilon_1 + \dots + \epsilon_i$ are the fundamental weights. Denote by ℓ and S a line and an l -dimensional subspace corresponding to points of G/P and G/Q . Denote by x_i and y_{i_1, \dots, i_l} the Plücker coordinates on G/P and G/Q . Denote by E_k a B -stable subspace $\langle e_1, \dots, e_k \rangle$.

Here is a list of B -stable prime divisors D_i (determined by geometric conditions on ℓ , S), their equations F_i in Plücker coordinates, degrees and weights of F_i :

D_1	$\ell \subset E_{n-1}$	x_n	$(1, 0)$	ω_1
D_2	$S \cap E_l \neq 0$	$y_{l+1, \dots, n}$	$(0, 1)$	ω_l
D_3	$(S + \ell) \cap E_{l-1} \neq 0$	$\sum_{i \geq l} (-1)^i x_i y_{l, \dots, \hat{i}, \dots, n}$	$(1, 1)$	ω_{l-1}
D_4	$(S + \ell) \cap \ell^\perp \cap E_l \neq 0$	$\sum_{i \leq l} (x_i y_{l+1, \dots, n} + \sum_{j > l} (-1)^{l+j} x_j y_{i, l+1, \dots, \hat{j}, \dots, n}) x_{n+1-i}$	$(2, 1)$	ω_l

The points in the complement of these divisors belong to the open B -orbit. Indeed, assume that a point does not belong to D_2 . Consider the matrix whose columns are the basis vectors of S . By choosing a basis we can assume that the lower $l \times l$ submatrix is the identity matrix. We can make other matrix entries equal to zero by the action of B . Suppose the point in addition does not belong to D_1 . Then the lowest entry of the column generating ℓ is nonzero. Now we can make entries of this column at positions $l+1, \dots, n-1$ equal to zero. Suppose the point

in addition does not belong to D_3 . Then the l -th entry of the column generating ℓ is nonzero. By the action of B we can make entries at positions $2, \dots, l-1$ equal to zero. Suppose the point does not belong to D_4 . Then the 1-st entry in the column is nonzero. By the action of B we can make these three nonzero entries equal to 1, i.e., now the point has the unique canonical form.

Up to a scalar multiple, B -semi-invariant functions are ratios of products of F_i such that degrees in every group of Plücker coordinates for the numerator and the denominator are equal. Hence we can find a lattice $\Lambda(X)$: it is generated by weights $\epsilon_1 - \epsilon_l$ and $\epsilon_1 + \epsilon_l$. We can take $f_{\epsilon_1 - \epsilon_l} = \frac{F_1 F_3}{F_4}$, $f_{\epsilon_1 + \epsilon_l} = \frac{F_1 F_2}{F_3}$ as basis weight functions. In the basis dual to the weights of these weight functions, the vectors $v_{D_1} = (1, 1)$, $v_{D_2} = (0, 1)$, $v_{D_3} = (1, -1)$, $v_{D_4} = (-1, 0)$ correspond to the divisors.

Any divisor δ is equivalent to a linear combination of the preimages of Schubert divisors: $\delta = pD_1 + qD_2$. The space of sections of the line bundle $\mathcal{O}(\delta)$ is the tensor product of the spaces of sections $\mathcal{O}(p\pi_1(D_1))$ and $\mathcal{O}(q\pi_2(D_2))$, where π_1 , π_2 are projections of X to G/P , G/Q and $\pi_1(D_1)$, $\pi_2(D_2)$ are Schubert divisors. The spaces of sections of $\mathcal{O}(p\pi_1(D_1))$ and $\mathcal{O}(q\pi_2(D_2))$ are isomorphic to $V_{p\omega_1}$ and $V_{q\omega_l}$, respectively. Thus for decomposing the product $V_{p\omega_1} \otimes V_{q\omega_l}$ it is sufficient to compute $H^0(G/P \times G/Q, \mathcal{O}(pD_1 + qD_2))$. The weight polytope is equal to $\mathcal{P}(\delta) = \{\lambda = -a\epsilon_1 - b\epsilon_l \mid 0 \leq b \leq a \leq p, a + b \leq 2q\}$. Using Theorem 3, we obtain a decomposition:

$$V_{p\omega_1} \otimes V_{q\omega_l} = \bigoplus_{\substack{0 \leq b \leq a \leq p \\ a+b \leq 2q \\ a \equiv b \pmod{2}}} V_{(p+q-a)\epsilon_1 + q\epsilon_2 + \dots + q\epsilon_{l-1} + (q-b)\epsilon_l}.$$

Example 2. Let $G = SL_n$. Consider the double flag variety corresponding to the pair $(3, p_2)$, (q_1, q_2, q_3) of parabolic subgroups. We assume that $q_1, q_2, q_3 \geq 3$. This is a variety of complexity 1. We use notation similar to notation from Example 1. Assume that $\omega_0 = \omega_n = 0$. Note that $\epsilon_i^* = -\epsilon_{n+1-i}$.

Denote by R_i and S_j the subspaces corresponding to points in G/P and G/Q , the lower index denotes the dimension of a subspace. Here is a list of B -stable prime divisors (determined by geometric conditions on R_i , S_j), degrees of their equations F_i in Plücker coordinates, and weights of F_i :

D_1	$R_3 \cap E_{n-3} \neq 0$	$(1, 0, 0)$	ω_3^*
D_2	$S_{q_1} \cap E_{n-q_1} \neq 0$	$(0, 1, 0)$	$\omega_{q_1}^*$
D_3	$S_{q_1+q_2} \cap E_{n-(q_1+q_2)} \neq 0$	$(0, 0, 1)$	$\omega_{q_1+q_2}^*$
$D_{4,5,6}$	$\langle R_3 \cap E_{n-3+k} + S_{q_1} \cap E_{n-3+k} \rangle \cap E_{n-q_1-k} \neq 0, \quad k = 1, 2, 3$	$(1, 1, 0)$	$\omega_{3-k}^* + \omega_{q_1+k}^*$
$D_{7,8,9}$	$\langle R_3 \cap E_{n-3+k} + S_{q_1+q_2} \cap E_{n-3+k} \rangle \cap E_{n-(q_1+q_2)-k} \neq 0, \quad k = 1, 2, 3$	$(1, 0, 1)$	$\omega_{3-k}^* + \omega_{(q_1+q_2)+k}^*$
D_{10}	$\langle \langle R_3 \cap E_{n-1} + S_{q_1} \cap E_{n-1} \rangle \cap E_{n-q_1-1} + S_{q_1+q_2} \cap E_{n-q_1-1} \rangle \cap E_{n-(q_1+q_2)-1} \neq 0$	$(1, 1, 1)$	$\omega_1^* + \omega_{q_1+1}^* + \omega_{(q_1+q_2)+1}^*$
D_{11}	$\langle \langle R_3 + S_{q_1} \rangle \cap E_{n-q_1-2} + S_{q_1+q_2} \cap E_{n-q_1-2} \rangle \cap E_{n-(q_1+q_2)-1} \neq 0$	$(1, 1, 1)$	$\omega_{q_1+2}^* + \omega_{(q_1+q_2)+1}^*$
D_{12}	$\langle \langle R_3 + S_{q_1} \rangle \cap E_{n-q_1-1} + S_{q_1+q_2} \cap E_{n-q_1-1} \rangle \cap E_{n-(q_1+q_2)-2} \neq 0$	$(1, 1, 1)$	$\omega_{q_1+1}^* + \omega_{(q_1+q_2)+2}^*$
D_{13}	$\langle R_3 \cap E_{n-2} + \langle R_3 + S_{q_1} \rangle \cap E_{n-q_1-2} + S_{q_1+q_2} \cap E_{n-q_1-2} \rangle \cap E_{n-(q_1+q_2)-2} \neq 0$	$(2, 1, 1)$	$\omega_2^* + \omega_{q_1+2}^* + \omega_{(q_1+q_2)+2}^*$
$D(z)$ $z \in \mathbb{P}^1$ $z \neq 0, 1, \infty$	$F_4 F_8 F_{11} - z F_5 F_7 F_{12}$	$(3, 2, 2)$	$\omega_1^* + \omega_2^* + \omega_{q_1+1}^* + \omega_{q_1+2}^* + \omega_{q_1+q_2+1}^* + \omega_{q_1+q_2+2}^*$

Consider polynomials in F_i , $i = 1, \dots, 13$, as polynomials in Plücker coordinates. The subspace of polynomials of weight $\omega_1^* + \omega_2^* + \omega_{q_1+1}^* + \omega_{q_1+2}^* + \omega_{q_1+q_2+1}^* + \omega_{q_1+q_2+2}^*$ and multidegree $(3, 2, 2)$ has dimension 2 and is linearly spanned by polynomials $F_4 F_8 F_{11}$, $F_5 F_7 F_{12}$, $F_{10} F_{13}$. Weight subspaces of lower degrees have dimension 1. These three polynomials are linearly dependent. Multiplying F_i by a scalar we may assume that the equation of linear dependence is $F_4 F_8 F_{11} - F_5 F_7 F_{12} + F_{10} F_{13} = 0$. We can regard the polynomials in this weight subspace as linear forms on \mathbb{P}^1 by taking $F_4 F_8 F_{11}$ and $F_5 F_7 F_{12}$ for homogeneous coordinates.

The valuation corresponding to a divisor $D(z)$ has order $h = 1$ and center at z . The vector $v_{D(z)}$ corresponding to this divisor equals zero. The valuations corresponding to the divisors D_4 , D_8 and D_{11} have order $h = 1$ and center at 0; the valuations corresponding to the divisors D_5 , D_7 and D_{12} have order $h = 1$ and center at ∞ ; the valuations corresponding to the divisors D_{10} and D_{13} have order $h = 1$ and center at 1. For other divisors D_i the corresponding valuations have order $h = 0$.

B -semi-invariant functions are constructed in the same way as in Example 1, but up to multiplication by a function from $\mathbb{C}(X)^B$. B -invariant functions are ratios of homogeneous polynomials of same degree in coordinates $F_4 F_8 F_{11}$ and $F_5 F_7 F_{12}$, i.e., the field $\mathbb{C}(X)^B$ is generated by the function $\frac{F_4 F_8 F_{11}}{F_5 F_7 F_{12}}$.

The lattice $\Lambda(X)$ is generated by the weights $\epsilon_i - \epsilon_j$, where i and j are numbers from different triples $(1, 2, 3)$, $(q_1 + 1, q_1 + 2, q_1 + 3)$, and $(q_1 + q_2 + 1, q_1 + q_2 + 2, q_1 + q_2 + 3)$.

We take the following functions as the basis weight functions:

$$\begin{aligned} \frac{F_4 F_{11}}{F_5 F_{10}} &= f_{\epsilon_2^* - \epsilon_1^*}, \frac{F_1 F_{11}}{F_5 F_7} = f_{\epsilon_3^* - \epsilon_1^*}, \frac{F_{12}}{F_2 F_8} = f_{\epsilon_{q_1+1}^* - \epsilon_1^*}, \frac{F_{11}}{F_{10}} = f_{\epsilon_{q_1+2}^* - \epsilon_1^*}, \frac{F_6}{F_5} = f_{\epsilon_{q_1+3}^* - \epsilon_1^*}, \\ \frac{F_{11}}{F_3 F_5} &= f_{\epsilon_{q_1+q_2+1}^* - \epsilon_1^*}, \frac{F_{12}}{F_{10}} = f_{\epsilon_{q_1+q_2+2}^* - \epsilon_1^*}, \frac{F_9}{F_8} = f_{\epsilon_{q_1+q_2+3}^* - \epsilon_1^*}. \end{aligned}$$

Let a_i be the coordinates of $\lambda \in \Lambda \otimes \mathbb{Q}$ in the basis of weights of the above B -semi-invariant functions. Let $\delta = m_1 D_1 + m_2 D_2 + m_3 D_3$. Then we obtain the following inequalities on coordinates defining the polytope $\mathcal{P}(\delta)$:

$$a_2 \geq -m_1, a_3 \leq m_2, a_6 \leq m_3, a_5 \geq 0, a_8 \geq 0,$$

and the following decomposition:

$$V_{m_1 \omega_3} \otimes V_{m_2 \omega_{q_1} + m_3 \omega_{q_1+q_2}} = \bigoplus m(\bar{a}) V_{\lambda(\bar{a}, \bar{m})},$$

where

$$\begin{aligned} m(\bar{a}) &= \max(0, 1 + \min(-a_1 - a_2 - a_5 - a_6, -a_2, a_3 + a_7) + \\ &\quad + \min(a_1, -a_3 - a_8, a_1 + a_2 + a_4 + a_6) + \min(-a_1 - a_4 - a_7, 0)), \end{aligned}$$

$$\begin{aligned} \lambda(\bar{a}, \bar{m}) &= m_1 \omega_3 + m_2 \omega_{q_1} + m_3 \omega_{q_1+q_2} - (a_1 + \dots + a_8) \epsilon_1 + a_1 \epsilon_2 + a_2 \epsilon_3 + \\ &\quad + a_3 \epsilon_{q_1+1} + a_4 \epsilon_{q_1+2} + a_5 \epsilon_{q_1+3} + a_6 \epsilon_{q_1+q_2+1} + a_7 \epsilon_{q_1+q_2+2} + a_8 \epsilon_{q_1+q_2+3}, \end{aligned}$$

and the sum ranges over all a_i that satisfy the inequalities given above.

3. SOME THEOREMS ABOUT COMPLEXITY OF DOUBLE FLAG VARIETIES

Now we formulate some theorems. The theorem we need for computing the complexity of a double flag variety is due to Panyushev:

Theorem 5 ([3]). *Suppose P and Q are decomposed into semidirect product of the standard Levi subgroup and the unipotent radical: $P = L \ltimes P_u$, $Q = M \ltimes Q_u$. Then the complexity of the action $G : G/P \times G/Q$ equals the complexity of the action $L \cap M : \mathfrak{p}_u \cap \mathfrak{q}_u$, where \mathfrak{p}_u and \mathfrak{q}_u are the Lie algebras of P_u and Q_u .*

Lemma 1. *The complexity does not change if we swap P and Q .*

Lemma 2. *Let $P' \subseteq P$, $Q' \subseteq Q$ be parabolic subgroups. Then the complexity of the double flag variety for the pair (P', Q') is not less than for the pair (P, Q) .*

Proof. There exists a G -equivariant surjective morphism $G/P' \times G/Q' \rightarrow G/P \times G/Q$. So the codimension of a general B -orbit on $G/P \times G/Q$ is not greater than the corresponding codimension on $G/P' \times G/Q'$. \square

Lemma 3. $c \geq \frac{1}{2}(\dim G - \dim L - \dim M - \dim T)$, where c is the complexity of the action.

Proof. It is easy to see that $c \geq \dim(\mathfrak{p}_u \cap \mathfrak{q}_u) - \dim(L \cap M \cap B)$. Besides, we have $\dim(L \cap M \cap B) = \frac{1}{2}(\dim(L \cap M) + \dim T)$, $\dim(\mathfrak{p}_u \cap \mathfrak{q}_u) = \frac{1}{2}(\dim G - \dim L - \dim M + \dim(L \cap M))$. Substituting these equalities in the first inequality, we prove the lemma. \square

4. CASE OF CLASSICAL MATRIX GROUPS

In this section G denotes SL_n , SO_n , or Sp_n . We assume that a Borel subgroup $B \subseteq G$ consists of upper-triangular matrices, that SO_n preserves the quadratic form with the matrix

$$\begin{pmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{pmatrix},$$

and that Sp_n preserves the skew-symmetric bilinear form with the matrix

$$\begin{pmatrix} 0 & & & & 1 \\ & \ddots & & & \\ & & 1 & & \\ & & -1 & & \\ -1 & & & \ddots & 0 \end{pmatrix}.$$

For computing the complexity we use Theorem 5. Now we describe Levi subgroups, Lie algebras of unipotent radicals and their intersections.

The Levi subgroup L (or M) consists of block-diagonal matrices; for groups SO_n and Sp_n the sizes of these blocks are symmetric with respect to the secondary diagonal and the matrices standing at symmetric places are A and $(A^S)^{-1}$ (here S denotes the transposition with respect to the secondary diagonal) and the central block (it exists if the number of blocks is odd) is an orthogonal or symplectic matrix respectively.

The Lie algebra \mathfrak{sl}_n consists of matrices with trace 0; the Lie algebra \mathfrak{so}_n in the chosen basis consists of matrices which are antisymmetric with respect to the secondary diagonal; the Lie algebra \mathfrak{sp}_n in the chosen basis consists of the following matrices: divide a matrix into 4 equal square parts, then the upper right part and the lower left part are symmetric with respect to the secondary diagonal, the other two parts are antisymmetric to each other with respect to the secondary diagonal. Matrices in the Lie algebra of the unipotent radical \mathfrak{p}_u (or \mathfrak{q}_u) have zeroes below the diagonal and in diagonal blocks.

For SO_n with even n there exists another class of parabolic subgroups (we call them *special*). We can obtain these subgroups from block-triangular parabolic subgroups without central diagonal block by conjugation with transposition of two middle basis vectors. We consider special parabolic subgroups separately.

Matrices from $L \cap M$ consist of several diagonal square blocks. We denote these blocks by A_1, \dots, A_r and their sizes by k_1, \dots, k_r . Besides, for SO_n and Sp_n we have a relation $k_i = k_{r+1-i}$. Note that for SO_n a middle pair of blocks of sizes 1 and 1 is the same as one middle block of size 2. Further, we assume that parabolic subgroups are not special. We consider the case of special subgroups separately. Matrices from $\mathfrak{p}_u \cap \mathfrak{q}_u$ consist of submatrices X_{ij} , where the matrix X_{ij} is of size $k_i \times k_j$ and stands at the intersection of rows passing through A_i and columns passing through A_j . Besides, $X_{ij} = 0$ if $i \geq j$ or if there exists a matrix from L or M with nonzero entries at the place of X_{ij} . By “blocks” we often mean *nonzero* matrices X_{ij} . A Borel subgroup in $L \cap M$ is $L \cap M \cap B$, i.e., the intersection of $L \cap M$ with upper triangular matrices. The group $L \cap M \cap B$ acts on $\mathfrak{p}_u \cap \mathfrak{q}_u$ by conjugation; matrices X_{ij} are transformed to $A_i X_{ij} A_j^{-1}$.

The idea is to consider all possible locations of blocks X_{ij} and to compute complexity for all sizes of blocks for each location. We need the following lemmas to simplify the case-by-case considerations and to reduce the number of possible cases.

Lemma 4. *The complexity does not change if we transpose simultaneously P and Q with respect to the secondary diagonal.*

Remark. This lemma gives simplification only for SL_n .

Lemma 5. *Consider an action, obtained from the original action by one of the following operations (or their combination):*

- remove some blocks X_{ij} (i.e., we assume that some X_{ij} are equal to 0),
- remove some matrices A_i and blocks X_{ij} in corresponding rows and columns.

Then the complexity for the new action cannot be greater than for the original one. In other words, we consider only “a part of an action”.

Proof. The first operation corresponds to the restriction of an action to a G -stable subspace. The complexity of the action on a G -stable subvariety cannot be greater than the complexity of the action on the initial variety [9].

The second operation corresponds to considering a quotient representation for which the complexity can be only less or equal than the original complexity. \square

Lemma 6. *Suppose there are 4 nonzero matrices X_{pq} standing at vertices of a rectangle, i.e., they have indices ij, il, kj and kl . We require that these matrices do not stand on the secondary diagonal for SO_n . Then there is a rational invariant for the action of $B \cap L \cap M$. We call this invariant the invariant of type “square”.*

Remark. The matrices from \mathfrak{so}_n have zeroes on the secondary diagonal. That is the reason why we have additional restriction on the positions of blocks for SO_n .

Proof. Suppose a_i, a_k are right lower entries of matrices A_i, A_k , a_j, a_l are left upper entries of matrices A_j, A_l , and $x_{ij}, x_{il}, x_{kj}, x_{kl}$ are left lower entries of matrices $X_{ij}, X_{il}, X_{kj}, X_{kl}$. Then $x_{pq} \rightarrow a_p x_{pq} a_q^{-1}$, $p = i, k, q = j, l$. It is easy to see that $x_{ij} x_{kj}^{-1} x_{kl} x_{il}^{-1}$ is an invariant. \square

Lemma 7. *Suppose there are 3 nonzero matrices X_{pq} standing in a special way at vertices of a rectangular triangle, i.e., they have indices ij, ik, jk . We require that these matrices do not stand on the secondary diagonal for SO_n . Then there is a rational invariant for the action of $B \cap L \cap M$. We call this invariant the invariant of type “triangle”.*

Proof. Suppose \bar{x}_{ij} is the lowest row of the matrix X_{ij} , x_{ik} is the left lowest entry of X_{ik} , and \bar{x}_{jk} is the left column of X_{jk} . It is easy to check that $\frac{\bar{x}_{ij} \cdot \bar{x}_{jk}}{x_{ik}}$ is an invariant. \square

Remark. The invariants of type “square” and “triangle” do not change for groups SO_n and Sp_n if we consider other blocks obtained by transposition with respect to the secondary diagonal.

Lemma 8. *Suppose there are 3 nonzero matrices X_{ij} in one row such that their height is at least 2. We require that these matrices do not stand on the secondary diagonal for group SO_n . Then the complexity is at least 1. If there are 4 such matrices, then the complexity is at least 2.*


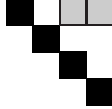
Proof. We prove the first statement for SL_n (since SO_n and Sp_n are subgroups of SL_n , we obtain this statement for other groups as a consequence). By the group action we can make left lower entry and the entry above of the first general matrix equal to 1 and 0 respectively. We do not want to change these entries further. Thus we can act on the left only by multiplication by a matrix such that its lower right 2×2 submatrix is diagonal. Then we can make the same entries of the second general matrix equal to 1. In order not to change these 4 entries we must act on the left only by matrices having λE as the lower right 2×2 submatrix. Now consider the same two entries of the third matrix (they are nonzero for a general matrix): they are multiplied by one and the same number. We can make one of them equal to 1 and after that we cannot change another one without changing other 5 considered entries. Thus general orbits depend at least on one continuous parameter, i.e., $c(X) \geq 1$. The proof for 4 matrices is similar. \square

Now we discuss a method for computing complexity of the action of $L \cap M$ on $\mathfrak{p}_u \cap \mathfrak{q}_u$. The Lie algebras of SO_n and Sp_n have symmetry in their block structure. So it is sufficient to consider the blocks on and below the secondary diagonal. By the action of $B \cap L \cap M$ we can put our blocks, one by one, in some canonical form and consider the action of the stabilizer of this canonical form on the remaining blocks. The number of the parameters left is the complexity. The same method was used in the proof of Lemma 8.

For SO_n with even n there are special parabolic subgroups, which we mark with strokes. We may assume that only one of the parabolic subgroups is special and the second one does not have a middle block (in the converse case we apply the automorphism of SO_n that transposes two middle basis vectors). We can estimate the complexity from below by the complexity of another action such that both parabolic subgroups are not special. For this, let us conjugate the special subgroup with transposition of two middle basis vectors and replace two middle blocks by one (here we may assume that the size of two middle blocks is not greater than the respective size for the second group). We enlarge the parabolic subgroup, so the complexity can only become smaller.

Now consider particular cases. The pictures show the location of nonzero blocks X_{ij} and matrices A_i ; the blocks X_{ij} are grey and the matrices A_i are black. We denote the complexity by c . We enumerate the possible locations of the blocks X_{ij} . If Lemmas 5, 6, 7, 8 give an estimate $c \geq 2$ for a given case, we shall not consider this case. We shall not consider cases, that are symmetrical to the cases already considered. The results of our considerations are presented below. We indicate only the cases in which the complexity is not greater than 1.

4.1. Group SL_n . 1.   k_1, k_r arbitrary $c = 0$

2.   $k_1 \leq 2$ k_{r-1}, k_r arbitrary $c = 0$
 k_1 arbitrary $k_{r-1} = 1$ or $k_r = 1$ $c = 0$
 $k_1 \geq 3$ $k_{r-1} = k_r = 2$ $c = 1$
 $k_1 = 3$ $k_{r-1}, k_r \geq 2$ $c = 1$

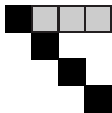
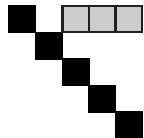
As an example, we consider this case in details. Without loss of generality we may assume that $k_{r-1} \geq k_r$. Acting from the right, we put two general blocks in the following form: the entries on the secondary diagonal coming from the lower left corner are equal to 1 and the entries to the right of this diagonal are equal to 0. We can make entries of the first block above the secondary diagonal coming from the left lower corner equal to 0. Let us find the stabilizer of this form. A_1 has zeroes in all entries except the diagonal and the left upper $\max(k_1 - k_{r-1}, 0) \times \max(k_1 - k_{r-1}, 0)$ submatrix. A_{r-1} and A_r have similar form, but the submatrix is lower right and of the size $\max(k_i - k_1, 0) \times \max(k_i - k_1, 0)$, where $i = r - 1, r$, respectively; the diagonal entries are equal to the diagonal entries of A_1 (in order to preserve 1's in blocks). Now we can make entries in the first column and rows $2, \dots, \min(k_1, k_{r-1})$ from the bottom of the second block equal to 1. Consider the stabilizer of this form. All diagonal entries of A_1 , A_{r-1} and A_r except entries in the considered submatrices are equal to one and the same number λ .

Suppose $k_1 \leq 2$ or $k_r = 1$. Then we can make entries in the first column of the second block equal to 1. Thus there are no free parameters, i.e., a general point lies in the orbit of the point of the described form, therefore $c = 0$.

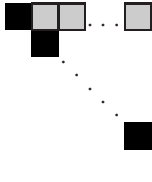
Suppose $k_1 = 3$, $k_{r-1} \geq k_r \geq 2$. Then we can make the entry in the first column and in the third row from the bottom of the second block equal to 1 (if it is not already 1). Then $A_1 = \lambda E$ and the two upper diagonal entries of A_r equal λ . Thus we cannot change the entry in the second column and in the third row from the bottom. Thus a general orbit depends on one continuous parameter, i.e., $c = 1$.

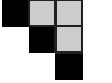
Now suppose $k_{r-1} = k_r = 2$, $k_1 \geq 4$. Consider the submatrix of the second block above the second row from the bottom. We can multiply it on the left by any upper triangular matrix and the action on the right reduces to multiplication of all entries by one and the same number. We can make the lower 2×2 submatrix of this submatrix equal to $\begin{pmatrix} 0 & 1 \\ 1 & * \end{pmatrix}$ and all entries above equal to zero. We cannot change the entry $*$, hence $c = 1$.

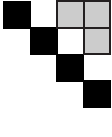
It remains to consider the case $k_1 \geq 4$, $k_{r-1} \geq 3$, $k_r \geq 2$. We can make the entry in the first column and in the fourth row from the bottom of the second block equal to 1 (if it is not already 1). Then the lower 4×4 submatrix of A_1 equals λE . Consider the entries in the second column and rows 3 and 4 from the bottom of the second block: we cannot change them. Thus $c \geq 2$.

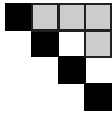
3.   $k_1 = 1$ k_{r-2}, k_{r-1}, k_r arbitrary $c = 0$
 $k_1 = 2$ k_{r-2}, k_{r-1}, k_r arbitrary $c = 1$
 $k_1 \geq 3$ $k_{r-2} = k_{r-1} = k_r = 1$ $c = 1$

If we add blocks to the first row, then their height cannot be greater than 1 by Lemma 8.

4.  $k_1 = 1 \quad c = 0$

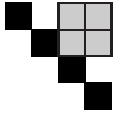
5a.  at least two of k_1, k_2, k_3 equal 1 $c = 1$

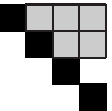
5b.  $k_1 = 1$ or $k_4 = 1$ k_2, k_3 arbitrary $c = 0$
 $k_1 = k_4 = 2$ k_2, k_3 arbitrary $c = 1$
 $k_1 = 2, k_4 \geq 3$ $k_2 = 1, k_3$ arbitrary $c = 1$
 $k_1 \geq 3, k_4 = 2$ k_2 arbitrary, $k_3 = 1$ $c = 1$

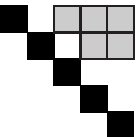
6.  $k_1 = 1 \quad k_2 = 1$ or $k_4 = 1 \quad c = 1$

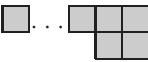
7. 

This case appears only when the number of blocks in the first row is $s \leq 3$. This is Case 5 or 6.

8.  $k_1 = k_2 = 1$ or $k_{r-1} = k_r = 1 \quad c = 1$

9a.  $c \geq 2$, because there are independent invariants of types “square” and “triangle”

9b.  $k_1 = 1 \quad k_2 = 1 \quad k_3$ arbitrary k_4, k_5 arbitrary $c = 1$
 $k_1 = 1 \quad k_2$ arbitrary k_3 arbitrary $k_4 = k_5 = 1 \quad c = 1$

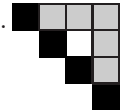
10. 

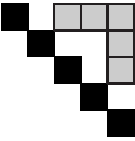
This case appears only when the number of blocks in the first row is $s \leq 4$.

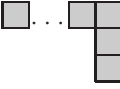
$s = 3$: this is Case 9.

$s = 4$: $c \geq 2$, because there are independent invariants of types “square” and “triangle”.

If there are at least 3 blocks in the second row, then there are two independent invariants of type “square”, i.e., $c \geq 2$.

11a.  $c \geq 2$, because there are two invariants of type “triangle”

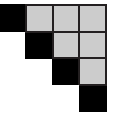
11b.  $k_1 = k_5 = 1 \quad c = 1$

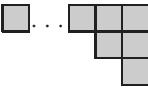
12. 

This case appears only when the number of blocks in the first row is $s \leq 4$.

$s = 3$: this is Case 11.

$s = 4$: $c \geq 2$, because there are two invariants of type “triangle”.

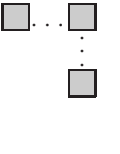
13.  $c \geq 2$ as there are invariants of types “square” and “triangle”

14. 

This case appears only when the number of blocks in the first row is $s \leq 4$.

$s = 3$: this is Case 13.

$s = 4$: $c \geq 2$, since this case can be reduced to Case 13 by Lemma 5

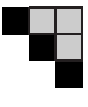
15. 

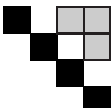
These cases appear only when $|s - m| \leq 1$ (here s, m are the numbers of blocks in the first row and in the last column, respectively). If $s, m \geq 4$ (these are the remaining cases), then the complexity is at least 2, because there are two invariants of type “triangle” (Case 15) or invariants of types “triangle” and “square” (Case 16).

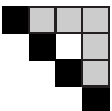
Combining all cases together we obtain the classification given in Table 1. Recall that the classification is given up to transposition with respect to the secondary diagonal and permutation of parabolics.

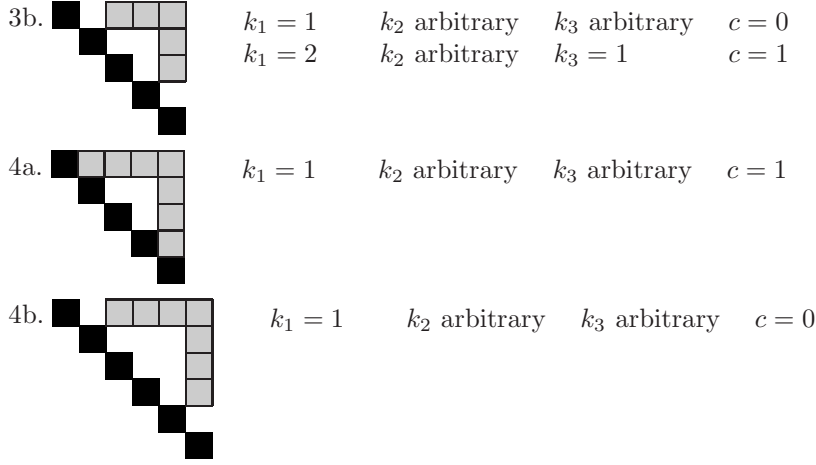
4.2. **Group \mathbf{SO}_n .** Consider the cases, where P and Q are not special.

1.  $c = 0$

2a.  $k_1 = 1 \quad k_2 \text{ arbitrary} \quad c = 0$
 $k_1 \text{ arbitrary} \quad k_2 = 1 \quad c = 0$
 $k_1 = 2 \quad k_2 = 2 \quad c = 1$

2b.  $k_1 \leq 3 \quad k_2 \text{ arbitrary} \quad c = 0$
 $k_1 \text{ arbitrary} \quad k_2 = 1 \quad c = 0$
 $k_1 = 4 \quad k_2 = 2 \quad c = 1$

3a.  $k_1 = 1 \quad k_2 \text{ arbitrary} \quad c = 0$
 $k_1 = 2 \quad k_2 = 1 \quad c = 1$



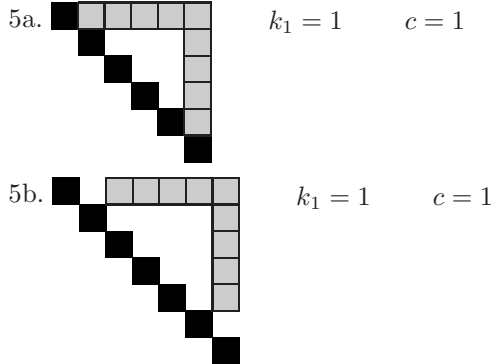
If there are at least 5 blocks in the first row, then by Lemma 8 their height cannot be greater than one.

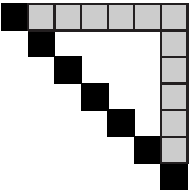
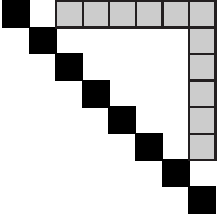
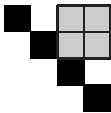
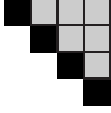
Suppose nonzero blocks stand only in the first row and in the last column (denote the number of blocks in the first row by m) and suppose that their height is $k_1 = 1$. Denote the complexity for this case by $c_{m,a}$ if $r = m + 1$ and by $c_{m,b}$ if $r = m + 2$ (here r is the number of diagonal blocks). Then we have the following lemma.

Lemma 9. *Suppose $m \geq 4$. We have $c_{m,a} = c_{m-1,b} + 1$, $c_{m,b} = c_{m-1,a}$.*

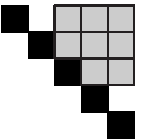
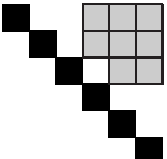
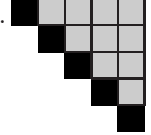
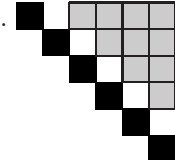
Proof. We can put a general pair of blocks $X_{1,i}$ and $X_{1,r+1-i}$ ($i \neq \frac{r+1}{2}$) to the form $(x, 0, \dots, 0)$ and $(t, 0, \dots, 0, y)$ by multiplication on the right if the widths of the blocks are at least 2, and to the form (x) and (y) if the widths are equal to 1. Here t, x are any nonzero numbers and y is determined by x , namely the product xy is invariant. A nonzero general block $X_{1, \frac{r+1}{2}}$ (if r is odd) can be put in the form $(x, 0 \dots 0, y)$ if the width of the block is at least 2, (where x and y are as above). If the width of this block equals 1 then we cannot change the entry in this block. All these matrices are multiplied on the left by one and the same number. Thus the complexity equals the number of these pairs plus the number of central blocks (0 or 1) minus 1. From this, we obtain the lemma. \square

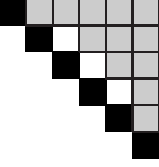
In Cases 5 and 6 we can easily compute complexities using this lemma, and $c \geq 2$ for $m \geq 7$.



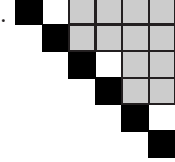
- 6a.  $k_1 = 1$ $c = 2$
- 6b.  $k_1 = 1$ $c = 1$
7.  $k_1 = 1$ k_2 arbitrary $c = 0$
 k_1 arbitrary $k_2 = 1$ $c = 0$
 $k_1 = 2$ $k_2 = 2$ $c = 1$
 $k_1 = 2$ $k_2 = 3$ $c = 1$
 $k_1 = 3$ $k_2 = 2$ $c = 1$
8.  $k_1 = 1$ $k_2 = 1$ $c = 0$
 $k_1 = 1$ $k_2 = 2$ $c = 1$
 $k_1 = 2$ $k_2 = 1$ $c = 1$

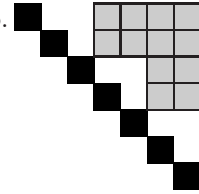
If we add blocks only to the first row (and add blocks symmetrical to them with respect to the secondary diagonal), then these cases do not appear.

- 9a.  $k_1 = 1$ $k_2 = 1$ $k_3 = 1$ $c = 1$
- 9b.  $k_1 = 1$ $k_2 = 1$ k_3 arbitrary $c = 0$
 $k_1 = 1$ $k_2 = 2$ $k_3 = 1$ $c = 1$
 $k_1 = 2$ $k_2 = 1$ $k_3 = 1$ $c = 1$
- 10a.  $c \geq 2$, because there are two invariants of type “triangle”
- 10b.  $k_1 = 1$ $k_2 = 1$ $k_3 = 1$ $c = 1$

11.  $c \geq 2$, because there are two invariants of type “triangle”

If we add blocks only to the first row (and add blocks symmetrical to them), then these cases do not appear.

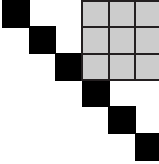
- 12a.  $c \geq 2$, because there are invariants of types “square” and “triangle”

- 12b.  $c \geq 2$, because there are invariants of types “square” and “triangle”

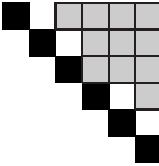
If we add one or two blocks to the first row (and add blocks symmetrical to them), then $c \geq 2$ (reduces to Case 12).

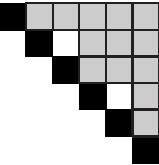
If we add more than two blocks to the first row (and add blocks symmetrical to them), then these cases do not appear.

If we add blocks to the second row (and to the first row, respectively, and add blocks symmetrical to them), then there are two invariants of type “square”.

13. 

$k_1 = 1$	$k_2 = 1$	$k_3 = 1$	$c = 0$
$k_1 = 1$	$k_2 = 1$	$k_3 = 2$	$c = 1$
$k_1 = 1$	$k_2 = 2$	$k_3 = 1$	$c = 1$
$k_1 = 2$	$k_2 = 1$	$k_3 = 1$	$c = 1$

14.  $k_1 = 1$ $k_2 = 1$ $k_3 = 1$ $c = 1$

15.  $c \geq 2$, because this case reduces to Case 11 by Lemma 5

If we add blocks to the first row (and add blocks symmetrical to them), then these cases do not appear.

If there are 3 blocks in the third row, 4 blocks in the second row, and 4 or 5 in the first row, then $c \geq 2$ (this follows from Case 12a). If there are ≥ 6 blocks in the first row, then these cases do not appear.


If there are ≥ 5 blocks in the second row, then there are two invariants of type “square”.


If there are ≥ 4 blocks in the third row then there are two invariants of type “square”.

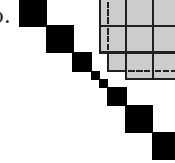
Now consider the case of special subgroups.

The numeration corresponds to the numeration of cases for non-special subgroups, obtained by replacing a special subgroup with a non-special one by conjugation with transposition of two middle basis vectors and by replacing two middle blocks with one central block. It is sufficient to consider the cases for which the size of two middle blocks obtained by this transformation is not greater than the size of two middle blocks for another subgroup.

0.  $c = 0$

2b.  $k_1 = 1$ k_2 arbitrary $c = 0$
 $k_1 = 2$ $k_2 = 1$ $c = 0$
 $k_1 = 2$ $k_2 = 2$ $c = 1$
 $k_1 = 3$ $k_2 = 1$ $c = 1$

3a.  $k_1 = 1$ $k_2 = 1$ $c = 1$


9b.  $k_1 = 1$ $k_2 = 1$ $k_3 = 1$ $c = 1$

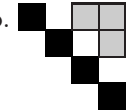
10b. If we want the complexity to be ≤ 1 , then it is necessary to have $k_3 = 1$. As one middle block of size 2 is the same as two middle blocks of sizes 1 and 1, then we may assume that in Case 10b (for non-special subgroups) both subgroups have two middle blocks, so we do not need to consider this case.

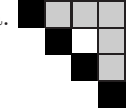
Combining all cases together we obtain the classification given in Table 2.

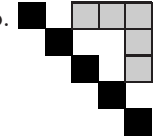
4.3. **Group \mathbf{Sp}_n .** The numbers of cases correspond to the numbers of cases for \mathbf{SO}_n .

1.  $c = 0$

2a.  $k_1 = 1$ k_2 arbitrary $c = 0$

2b.  $k_1 = 1$ k_2 arbitrary $c = 0$
 $k_1 = 2$ k_2 arbitrary $c = 1$

3a.  $k_1 = 1$ k_2 arbitrary $c = 1$

3b.  $k_1 = 1$ k_2 arbitrary k_3 arbitrary $c = 0$

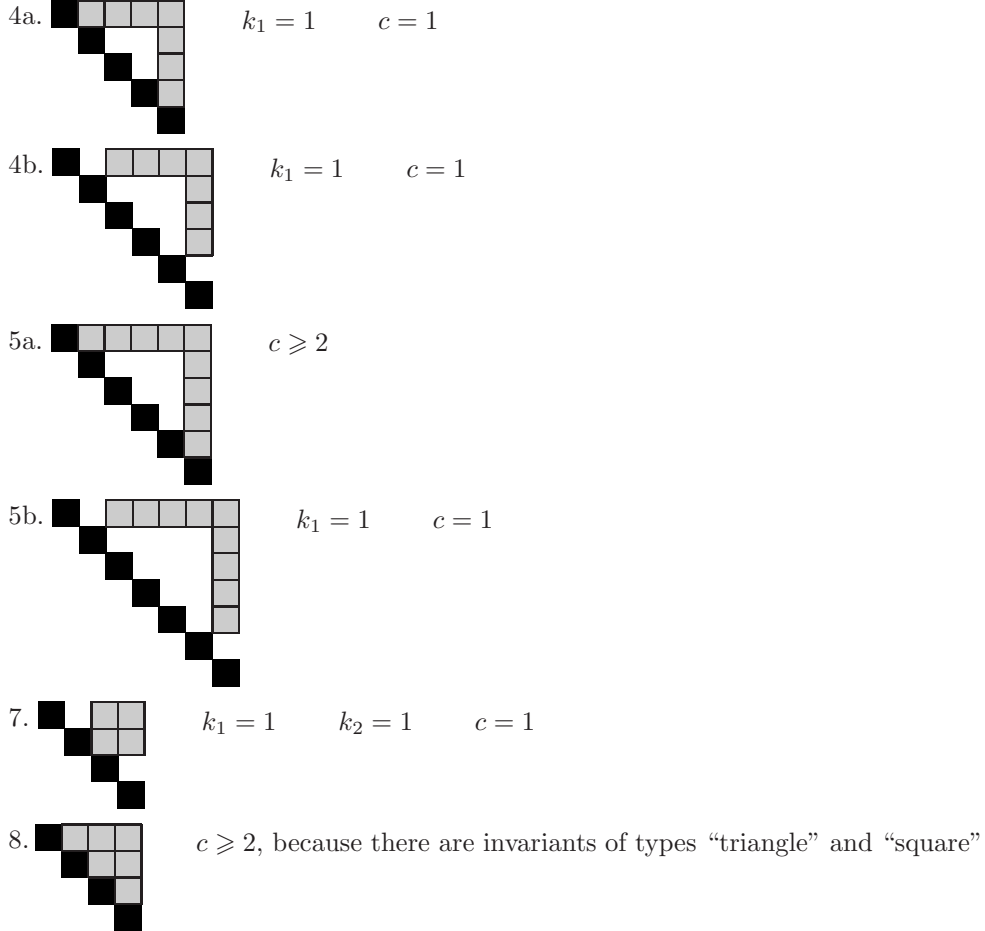
If there are at least 4 blocks in the first row, then by Lemma 8 their height cannot be greater than one.

Suppose nonzero blocks stand only in the first row and in the last column (denote the number of blocks in the first row by m) and suppose that their height is $k_1 = 1$. Denote the complexity for this case by $c_{m,a}$ if $r = m + 1$ and by $c_{m,b}$ if $r = m + 2$ (here r is the number of diagonal blocks). Then we have the following lemma.

Lemma 10. *Suppose $m \geq 3$. We have $c_{m,a} = c_{m-1,b} + 1$, $c_{m,b} = c_{m-1,a}$.*

The proof is similar to the proof of Lemma 10.

In Cases 4 and 5 we can easily compute complexities using this lemma, and $c \geq 2$ for $m \geq 6$.



If we add blocks to the first row (and add blocks symmetrical to them), then these cases do not appear.

If there are ≥ 3 blocks in the second row, then $c \geq 2$, because there are 2 invariants of type “square”.

Combining all cases together we obtain the classification given in Table 3.

5. CASE OF EXCEPTIONAL GROUPS

Fix a Borel subgroup $B \subseteq G$ and a maximal torus $T \subseteq B$. Suppose Δ is the system of roots with respect to T , Π is the system of simple roots corresponding to the choice of B , $I \subseteq \Pi$ is a subset. Any parabolic subgroup containing B coincides with a standard parabolic subgroup P_I whose Lie algebra can be decomposed into the direct sum of the Lie algebra of T and the root subspaces corresponding to positive roots and roots that are linear combinations with integer coefficients of

roots from I , i.e.,

$$\mathfrak{p}_I = \mathfrak{t} \oplus \bigoplus_{\{\alpha > 0\} \cup \{\alpha \in \mathbb{Z}I\}} \mathfrak{g}_\alpha$$

The Lie algebra \mathfrak{p}_I can be decomposed into the direct sum of the standard Levi subalgebra \mathfrak{l} and the Lie algebra of the unipotent radical. The roots from $\mathbb{Z}I$ correspond to the Lie subalgebra \mathfrak{l} and the other roots $\{\alpha > 0\} \cap \{\alpha \notin \mathbb{Z}I\}$ correspond to the unipotent radical.

Suppose $P = P_I = L \ltimes P_u$ and $Q = P_J = M \ltimes Q_u$ are two parabolic subgroups. Then we have

$$\begin{aligned} \mathfrak{l} \cap \mathfrak{m} &= \mathfrak{t} \oplus \bigoplus_{\alpha \in \mathbb{Z}(I \cap J)} \mathfrak{g}_\alpha, \\ \mathfrak{p}_u \cap \mathfrak{q}_u &= \bigoplus_{\alpha > 0, \alpha \notin \mathbb{Z}I \cup \mathbb{Z}J} \mathfrak{g}_\alpha. \end{aligned}$$

Now we describe a general method of computing the complexity of a linear representation of a reductive group [10, section 1.4]. Suppose G is a reductive group, V is its linear representation. Denote by v_λ a vector of weight λ . Consider a lowest weight vector $v_{-\lambda^*}$ of V . We can decompose the space V as follows: $V = \langle v_{-\lambda^*} \rangle \oplus W$, where W is B -stable. Consider an open B -stable subset $\mathring{V} = \mathbb{C}^\times v_{-\lambda^*} \oplus W$ in V . Let P be the parabolic subgroup preserving the line $\langle v_{-\lambda^*} \rangle \subseteq V^*$. Decompose P into a semidirect product of the Levi subgroup and the unipotent radical: $P = L \ltimes P_u$. Let V' be an L -stable complementary subspace to $\mathfrak{p}_u v_{-\lambda^*}$ in W , i.e., $\mathfrak{p}_u v_{-\lambda^*} \oplus V' = W$. The subset \mathring{V} is isomorphic to the direct product $\mathring{V} = P_u \times (\mathbb{C}^\times v_{-\lambda^*} \oplus V')$ as a B -variety. On the right-hand side, P_u acts on the first factor by left translations, and $B \cap L$ acts on the first factor by conjugation, while the action on the second factor is induced from the action of L . Therefore the codimension of a general orbit for the action $B = (B \cap L) \ltimes P_u : \mathring{V}$ equals the codimension of a general orbit for the action $B \cap L : \langle v_{-\lambda^*} \rangle \oplus V'$. Thus we reduced the question about the complexity of the action $G : V$ to the complexity of a smaller group L acting on a smaller space $\langle v_{-\lambda^*} \rangle \oplus V'$. In this way we can construct a sequence of groups $L^{(i)}$ and spaces $V^{(i)}$:

$$G = L^{(0)} \supseteq L^{(1)} \supseteq \dots \supseteq L^{(s)}$$

$$V = V^{(0)} \supseteq V^{(1)} \supseteq \dots \supseteq V^{(s)}$$

such that all irreducible $L^{(s)}$ -submodules $V^{(s)}$ are one-dimensional. Then the action $L^{(s)}$ on $V^{(s)}$ is determined by the weights μ_1, \dots, μ_N and the complexity equals $\dim V^{(s)} - \text{rk} \langle \mu_1, \dots, \mu_N \rangle = N - \text{rk} \langle \mu_1, \dots, \mu_N \rangle$.

Now we explain how this method is applied in our case. The intersection of the Levi subgroups $L \cap M$ and the intersection of the Lie algebras of the unipotent radicals $\mathfrak{p}_u \cap \mathfrak{q}_u$ are determined by some subsets of roots E_1 and F_1 corresponding to the weight subspaces of Lie algebras with nonzero weights. Suppose μ_1 is a minimal root from F_1 . Let $E'_1 = \{\alpha \in E_1 \mid \alpha + \mu_1 \in F_1\}$, $F'_1 = \{\alpha + \mu_1 \mid \alpha \in E'_1\}$. Put $E_2 = E_1 \setminus E'_1$, $F_2 = F_1 \setminus (F'_1 \cup \{\mu_1\})$. In the same way we construct μ_2 and E_3, F_3 for E_2 and F_2 and so on, while F_i is nonempty. So we obtain a set of weights $\mu_1, \mu_2, \dots, \mu_N$ and complexity equals $N - \text{rk} \langle \mu_1, \dots, \mu_N \rangle$.

For every exceptional group G we first compute complexity of double flag varieties for maximal parabolic subgroups (they correspond to subsets of simple roots obtained from the set of simple roots by removing one root). Then we reduce the parabolic subgroups. It is clear that we do not need to compute complexity for the cases where we have an estimate $c \geq 2$ from Lemmas 2, 3.

Now consider particular groups.

5.1. **G₂, F₄**. For these groups there are no pairs of maximal parabolic subgroups for which the estimate on complexity is ≤ 1 .

5.2. **E₈**. For all pairs of parabolic subgroups except the pair corresponding to the pair of roots (α_1, α_1) we have an estimate on complexity $c \geq 2$. For this pair the complexity equals 2. So there are no suitable cases.

5.3. **E₆, E₇**. The list of pairs of roots for which the estimate on complexity for corresponding parabolics is not greater than 1, and complexities are given in Table 4.

E_6			E_7		
$\Pi \setminus I$	$\Pi \setminus J$	complexity	$\Pi \setminus I$	$\Pi \setminus J$	complexity
1	1	0	1	1	0
1	2	0	1	2	1
1	4	0	1	6	0
1	5	0	1	7	0
1	6	0	6	6	2
2	5	0	1	1, 2	2
4	5	0	1	1, 6	2
5	5	0			
5	6	0			
6	6	2			
1	1, 2	1			
1	1, 5	0			
1	1, 6	1			
1	4, 5	1			
1	5, 6	1			
5	1, 2	1			
5	1, 5	0			
5	1, 6	1			
5	4, 5	1			
5	5, 6	1			

TABLE 4. pairs of parabolic subgroups such that the estimate on complexity is ≤ 1 , and corresponding complexities

For example we consider in details the case for the group E_6 and subsets $\Pi \setminus I = \{\alpha_1\}$, $\Pi \setminus J = \{\alpha_5\}$. In this case, in the notation of [4, Table 1], $E_1 = \{\varepsilon_i - \varepsilon_j \mid i, j = 2, \dots, 5; i < j\} \cup \{\varepsilon_6 + \varepsilon_i + \varepsilon_j + \varepsilon \mid i, j = 2, \dots, 5; i < j\}$, $F_1 = \{\varepsilon_1 - \varepsilon_6\} \cup \{\varepsilon_1 + \varepsilon_i + \varepsilon_j + \varepsilon \mid i, j = 2, \dots, 5; i < j\} \cup \{2\varepsilon\}$. Take $\varepsilon_1 - \varepsilon_6$ as μ_1 . Then $E'_1 = \{\varepsilon_6 + \varepsilon_i + \varepsilon_j + \varepsilon \mid i, j = 2, \dots, 5; i < j\}$ and $F'_1 = \{\varepsilon_1 + \varepsilon_i + \varepsilon_j + \varepsilon \mid i, j = 2, \dots, 5; i < j\}$, whence we have $E_2 = \{\varepsilon_i - \varepsilon_j \mid i, j = 2, \dots, 5; i < j\}$, and $F_2 = \{2\varepsilon\}$. Take 2ε as μ_2 , then $F_3 = \emptyset$. The complexity equals $2 - \text{rk}\langle \mu_1, \mu_2 \rangle = 0$.

The final result is formulated in Theorem 2.

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